

Boundary action of the H_3^+ model

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ABSTRACT: We find the boundary action for Euclidean AdS_2 D-branes in H_3^+ . This action is consistent with the D-branes' symmetries and with the H_3^+ -Liouville relation for disc correlators. It can be used for performing free-field calculations in the H_3^+ model with boundaries. We explicitly perform the Coulomb-like integrals which appear in the free-field calculation of the bulk one-point function, and find agreement with previously known conformal bootstrap results.

KEYWORDS: Conformal Field Models in String Theory, D-branes.

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1. Introduction and summary

The H_3^+ model on a sphere was solved ten years ago thanks to the methods of the conformal bootstrap [1], which rely on symmetry and consistency assumptions and do not exploit the Lagrangian definition of the model. It was later realized that concordant information on the structure constants of the H_3^+ model could independently be derived thanks to the so-called free-field approach [2–4], which consists of perturbative calculations based on the Lagrangian definition. Then, after the AdS_2 D-branes in AdS_3 [5], Euclidean AdS_2 D-branes were discovered in H_3^+ [6, 7], which is the Euclidean version of AdS_3 . The worldsheet description of strings ending on such D-branes is the H_3^+ model on a disc with maximally symmetric boundary conditions. The solution of this model by conformal bootstrap methods was recently completed [8, 9].

These developments have left the problem of the Lagrangian definition of the H_3^+ model on the disc open. In other words, what is the boundary action for the Euclidean AdS_2 D-branes? This action may be useful for obtaining a more synthetic perspective on the model, relating it to other models, and solving it on higher genus Riemann surfaces with boundaries. Path-integral calculations were indeed very helpful in the recent study of the H_3^+ model on higher genus closed Riemann surfaces [10].

The problem of finding the boundary action was addressed by Ponsot and Silva [11], who showed that the variations of the bulk H_3^+ action already vanished by themselves in the presence of AdS_2 D-branes. They concluded that the boundary action vanished. However, AdS_2 D-branes come in a family with a continuous parameter c , and the results of path-integral calculations should depend on c . One could try to impose the c -dependent gluing conditions as constraints on the path integral, but it is not clear how to compute the resulting constrained integral. Here, we will instead propose a boundary action (2.8) which vanishes on-shell but nevertheless contributes to path-integral calculations. We will show that the expected boundary conditions can be derived from this action.

We will argue that the path-integral expressions of disc correlators which follow from our boundary action agree with the known disc correlators. The argument relies on the simple relation of the known disc correlators in the H_3^+ model with disc correlators in Liouville theory [8]. It was recently shown that the relation between H_3^+ and Liouville correlators on a sphere could easily be derived by a formal path-integral calculation [10]. We will sketch a similar calculation in the case of correlators on a disc.

We will also argue that the boundary action can be used for performing free-field calculations. We will indeed first check that the boundary action preserves the expected current-algebra symmetries. We will then compute the bulk one-point function and find agreement with the conformal bootstrap result of [6, 7]. This free-field computation will involve the explicit determination of a family of bulk-boundary Coulomb-like integrals (5.6). We will actually only use particular cases of such integrals; the most general integrals would appear in free-field calculations of bulk-boundary two-point functions in Liouville theory and in the H_3^+ model.

We are informed that a similar setup for the boundary dynamics of the H_3^+ model was found by T. Creutzig and V. Schomerus in connection with their work on the $GL(1|1)$ supergroup WZNW model [12].

The basic concepts of boundary conformal field theory and non-rational conformal field theory which we will use are explained in the review articles [13, 14].

2. Classical analysis of the H_3^+ model with a boundary

Let us first define the bulk H_3^+ model on the complex plane, which is conformally equivalent to the Riemann sphere. We will parametrize the plane with a complex variable $z = \tau + i\sigma$ and denote $\iint \equiv \int d^2z$; the single integration symbol \int will be reserved for integrals over the boundary $z = \bar{z}$ of the upper half-plane. The model is defined by the following action, where we adopt the notations of [10] (while adding a “bulk cosmological constant” numerical factor λ to the interaction term)

$$S^{\text{bulk}} = \frac{1}{2\pi} \iint \left(\partial\phi\bar{\partial}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - \lambda b^2\beta\bar{\beta}e^{2b\phi} \right). \quad (2.1)$$

All fields are bosonic. The field γ has conformal dimension zero, the holomorphic field β has conformal dimension one, and ϕ has conformal dimension zero but a background charge b , so that the interaction term $\iint\beta\bar{\beta}e^{2b\phi}$ is conformally invariant with respect to the holomorphic stress-energy tensor

$$T = -\beta\partial\gamma - \partial\phi^2 + b\partial^2\phi. \quad (2.2)$$

Here $b > 0$ is a continuous parameter of the H_3^+ model, which is related to the level k and the central charge c by

$$b^2 = \frac{1}{k-2}, \quad c = \frac{3k}{k-2}. \quad (2.3)$$

The model actually has not only a conformal symmetry, but also an affine symmetry generated by currents which we denote as

$$\begin{aligned}
 J^- &= \beta, & \bar{J}^- &= \bar{\beta}, \\
 J^3 &= \beta\gamma + b^{-1}\partial\phi, & \bar{J}^3 &= \bar{\beta}\bar{\gamma} + b^{-1}\bar{\partial}\phi, \\
 J^+ &= \beta\gamma^2 + 2b^{-1}\gamma\partial\phi - k\partial\gamma, & \bar{J}^+ &= \bar{\beta}\bar{\gamma}^2 + 2b^{-1}\bar{\gamma}\bar{\partial}\phi - k\bar{\partial}\bar{\gamma}.
 \end{aligned}
 \tag{2.4}$$

Euclidean AdS_2 D-branes are maximally symmetric in that they preserve half of these six currents. Strings ending on such D-branes are described by the H_3^+ model on the complex upper half-plane, with the following gluing conditions at $z = \bar{z}$ [6, 11]:

$$\begin{aligned}
 J^- + \bar{J}^- &= 0, \\
 J^3 - \bar{J}^3 &= 0, \\
 J^+ + \bar{J}^+ &= 0.
 \end{aligned}
 \tag{2.5}$$

There is actually a one-parameter family of D-branes which satisfy such gluing conditions; namely, for any real number c , we can assume that at $z = \bar{z}$

$$\begin{aligned}
 \beta + \bar{\beta} &= 0, \\
 \gamma + \bar{\gamma} &= ce^{b\phi}, \\
 (\bar{\partial} - \partial)\phi &= cb\beta e^{b\phi}.
 \end{aligned}
 \tag{2.6}$$

To these gluing conditions, one may add the bulk equations of motion for the $\beta, \bar{\beta}$ fields, which have no reason to fail at the boundary $z = \bar{z}$:

$$\partial\bar{\gamma} = \lambda b^2 \beta e^{2b\phi}, \quad \bar{\partial}\gamma = \lambda b^2 \bar{\beta} e^{2b\phi}.
 \tag{2.7}$$

Modulo these equations, our gluing conditions are equivalent to the known gluing conditions on $(\phi, \gamma, \bar{\gamma})$ [11].

Let us clarify a subtlety about the derivation of the gluing conditions (2.5) for the currents (2.4) from the gluing conditions (2.6) for the fields (ϕ, β, γ) . This derivation should take into account the nature of (ϕ, β, γ) as quantum fields, so that the currents (2.4) involve regularized products of these fields. The regularization involves the explicit subtraction of the singularities in the operator products; these singularities can be deduced from eq. (4.3). Alternatively, we could do the same calculation in a classical framework which would treat the fields as ordinary functions on the worldsheet; but in this framework the expressions for the currents J^+, \bar{J}^+ differ from our formula (2.4). The classical version of J^+ is indeed $J_{cl}^+ = \beta\gamma^2 + 2b^{-1}\gamma\partial\phi - (k-2)\partial\gamma$, where the difference $J_{cl}^+ - J^+ = 2\partial\gamma$ arises from regularizing the operator product $\beta\gamma^2$.¹

Now consider the bulk action S^{bulk} (2.1) on the upper half-plane. Cancelling the variations of this action implies the bulk equations of motion, plus some constraints on the behaviour of the fields at the boundary $z = \bar{z}$. It was found in [11] that these constraints are satisfied by the gluing conditions (2.6) for all values of c . (The calculations in [11] were actually performed using the equivalent action obtained by integrating out the fields $\beta, \bar{\beta}$

¹We are very grateful to the JHEP referee for helping us clarify this point.

in the path integral.) However, the bulk action by itself cannot be enough for defining the quantum dynamics of the model, because it does not know the value of c . It is however still possible to add a boundary term to the action provided it vanishes when the gluing conditions (2.6) are obeyed, and we propose

$$S^{\text{bdy}} = \frac{i}{4\pi} \int \beta \left(\gamma + \bar{\gamma} - ce^{b\phi} \right), \quad \beta + \bar{\beta} \Big|_{z \rightarrow \bar{z}} = 0, \quad (2.8)$$

where the single integral $\int = \int d\tau$ means the integral over the boundary $z = \bar{z}$ of the upper half-plane. We therefore propose that in path-integral calculations the first gluing condition is imposed as a constraint, while the last two should follow from the variational principle applied to the action $S = S^{\text{bulk}} + S^{\text{bdy}}$.

Let us now study the variations of the action. Using an integration by parts and $\partial = \frac{1}{2}(\partial_\tau - i\partial_\sigma)$, the action is rewritten as

$$S = S^{\text{bulk}} + S^{\text{bdy}} = \frac{1}{2\pi} \iint \left(\partial\phi\bar{\partial}\phi - \gamma\bar{\partial}\beta - \bar{\gamma}\partial\bar{\beta} - \lambda b^2\beta\bar{\beta}e^{2b\phi} \right) - c\frac{i}{4\pi} \int \beta e^{b\phi}. \quad (2.9)$$

Taking into account the constraint $\delta(\beta + \bar{\beta}) \Big|_{z \rightarrow \bar{z}} = 0$, the boundary terms in the variations of the action are therefore

$$(\delta S)^{\text{bdy}} = \frac{i}{4\pi} \int \left[-\delta\phi \left((\partial - \bar{\partial})\phi + cb\beta e^{b\phi} \right) + \delta\beta \left(\gamma + \bar{\gamma} - ce^{b\phi} \right) \right]. \quad (2.10)$$

Requiring the vanishing of the coefficients of the independent variations $\delta\phi$ and $\delta\beta$ therefore yields the last two gluing conditions in eq. (2.6). This shows that our proposal for the boundary action is classically sound. Not only it does not spoil the compatibility of the desired gluing conditions with the variational principle, but also it singles out a value for the parameter c .

3. Path-integral derivation of the relation with Liouville theory

Let us now consider the path-integral representation of a general H_3^+ correlator on the upper half-plane (which is conformally equivalent to the disc), with a number of bulk and boundary operator insertions. We will show how this correlator is related to a Liouville theory correlator by integrating out the fields $\gamma, \bar{\gamma}$ and then $\beta, \bar{\beta}$. The calculation follows closely that of Hikida and Schomerus in the case of the sphere [10], so we will only sketch the few most relevant points.

The relevant bulk and boundary operators, with spins j and ℓ , isospins μ and ν and worldsheet positions z and τ respectively, are

$$\Phi^j(\mu|z) = |\mu|^{2j+2} e^{\mu\gamma(z) - \bar{\mu}\bar{\gamma}(\bar{z})} e^{2b(j+1)\phi(z, \bar{z})}, \quad (3.1)$$

$$\Psi^\ell(\nu|\tau) = |\nu|^{\ell+1} e^{\frac{1}{2}\nu(\gamma(\tau) - \bar{\gamma}(\tau))} e^{b(\ell+1)\phi(\tau)}. \quad (3.2)$$

The correlator to be computed is

$$\Omega = \int \mathcal{D}\phi \mathcal{D}\beta \mathcal{D}\bar{\beta} \mathcal{D}\gamma \mathcal{D}\bar{\gamma} e^{-S} \prod_{i=1}^n \Phi^{j_i}(\mu_i|z_i) \prod_{a=1}^m \Psi^{\ell_a}(\nu_a|\tau_a), \quad (3.3)$$

where the constraint $\beta + \bar{\beta} \underset{z \rightarrow \bar{z}}{=} 0$ is implicitly understood, and the action S is given in eq. (2.9), where the boundary parameter c can actually jump at the insertion points of boundary operators. Note that the sign of the action is such that the Gaussian integral over $\beta, \bar{\beta}$ is convergent provided β and $\bar{\beta}$ are complex anticonjugates. Integrating out $\beta, \bar{\beta}$ would produce the well-known H_3^+ sigma model [15], plus an extra boundary action.

As an aside, recall that large k limits of H_3^+ correlators can then be determined thanks to so-called minisuperspace computations. In such computations, the functional integrals $\int \mathcal{D}\phi \mathcal{D}\gamma \mathcal{D}\bar{\gamma}$ are replaced with ordinary integrals over the zero-modes $\int d\phi d^2\gamma$. Due to factors $\partial\bar{\gamma}$ or $\bar{\partial}\gamma$, the bulk and boundary interaction terms then vanish. This provides an a posteriori justification for the minisuperspace calculations of the bulk one-point function [6], bulk-boundary two-point function [8], and boundary three-point function [9], which did not involve any contributions from the then-unknown boundary action. (The results of such minisuperspace calculations nevertheless depend on the boundary parameter c because the zero-modes have to be integrated only over the D-brane's world-volume.)

Let us perform the path integral over $\gamma, \bar{\gamma}$. This yields delta-function constraints on derivatives of the fields $\beta, \bar{\beta}$, namely

$$\Omega = \int \mathcal{D}\phi \mathcal{D}\beta \mathcal{D}\bar{\beta} \delta \left(\frac{1}{2\pi} \bar{\partial}\beta(z) - \sum_i \mu_i \delta^{(2)}(z - z_i) - \frac{1}{2} \sum_a \nu_a \delta^{(2)}(z - \tau_a) \right) \times \delta \left(\frac{1}{2\pi} \partial\bar{\beta}(\bar{z}) + \sum_i \bar{\mu}_i \delta^{(2)}(z - \bar{z}_i) + \frac{1}{2} \sum_a \nu_a \delta^{(2)}(z - \tau_a) \right) \times \dots \quad (3.4)$$

Now, performing the path integral over $\beta, \bar{\beta}$ (subject to $\beta + \bar{\beta} \underset{z \rightarrow \bar{z}}{=} 0$) will yield a nonzero result only provided $\sum_i (\mu_i + \bar{\mu}_i) + \sum_a \nu_a = 0$, and will force $\beta, \bar{\beta}$ to adopt the values

$$\beta_s(z) = \sum_i \frac{\mu_i}{z - z_i} + \sum_i \frac{\bar{\mu}_i}{z - \bar{z}_i} + \sum_a \frac{\nu_a}{z - \tau_a}, \quad (3.5)$$

$$\bar{\beta}_s(\bar{z}) = - \sum_i \frac{\mu_i}{\bar{z} - z_i} - \sum_i \frac{\bar{\mu}_i}{\bar{z} - \bar{z}_i} - \sum_a \frac{\nu_a}{\bar{z} - \tau_a}. \quad (3.6)$$

Notice that the $\sum_i \frac{\bar{\mu}_i}{z - \bar{z}_i}$ terms of $\beta(z)$ do not contribute to $\bar{\partial}\beta(z)$, which is defined only in the upper half-plane whereas \bar{z}_i belong to the lower half-plane. However, such terms are required by the assumed condition $\beta + \bar{\beta} \underset{z \rightarrow \bar{z}}{=} 0$. Note also the subtlety in defining $\delta^{(2)}(z - \tau_a)$ when τ_a belongs to the boundary; the correct treatment of this subtlety (for instance by slightly moving τ_a into the upper half-plane) leads to the correct numerical factor of the term $\sum_a \frac{\nu_a}{z - \tau_a}$ in $\beta_s(z)$.

After replacing the fields $\beta, \bar{\beta}$ by their values $\beta_s, \bar{\beta}_s$ in the path integral, we should relate ϕ to the Liouville field so that Ω can be interpreted as a Liouville theory correlator (plus some simple factors). This is achieved by performing a change of variable on the field ϕ so that $\beta_s \bar{\beta}_s e^{2b\phi} = -e^{2b\tilde{\phi}}$, where $\tilde{\phi}$ is the Liouville field. In the case of the sphere, the effect of this change of variable on the kinetic term $\iint \partial\phi \bar{\partial}\phi$ could be interpreted as the introduction of degenerate Liouville operators at the zeroes of β_s , and the situation is the same in our case of the disc. We refer to [10] for the details. The new feature in our case is

the presence of the boundary term $-c\frac{i}{4\pi}\int\beta_s e^{b\phi}$. The change of variable $\phi\rightarrow\tilde{\phi}$ will only absorb β_s into the exponential up to an overall sign:

$$-c\frac{i}{4\pi}\int\beta_s e^{b\phi} = -c\frac{i}{4\pi}\int(\text{sgn}\beta_s) e^{b\tilde{\phi}}. \tag{3.7}$$

The value of the Liouville boundary cosmological constant (i.e. of the coefficient of $\int e^{b\tilde{\phi}}$) is therefore

$$\mu_B = -c\frac{i}{4\pi}\text{sgn}\beta_s \tag{3.8}$$

This relation between the H_3^+ and Liouville boundary parameters, and the rest of the H_3^+ -Liouville relation on the disc whose derivation we just sketched, fully agree with the known H_3^+ -Liouville relation on the disc [8], which was originally derived by conformal bootstrap methods. In particular, μ_B is pure imaginary for physical (i.e. real) values of c , and its sign is determined by the sign of β_s . This agreement amounts to an additional heuristic argument in favour of our boundary action S^{bdy} (2.8).

4. Free-field formalism

The total action S (2.9) can be split into free terms, plus bulk and boundary interaction terms corresponding to the Lagrangians

$$L^{\text{bulk}} = \beta\bar{\beta}e^{2b\phi}, \quad L^{\text{bdy}} = \beta e^{b\phi}. \tag{4.1}$$

The free theory is subject to the simple gluing conditions (which coincide with the gluing conditions (2.6) at $c=0$)

$$\begin{aligned} \beta + \bar{\beta} &= 0, \\ \gamma + \bar{\gamma} &= 0, \\ (\partial - \bar{\partial})\phi &= 0. \end{aligned} \tag{4.2}$$

The non-vanishing pairings of the basic fields in the presence of such gluing conditions are

$$\begin{aligned} \langle\phi(z)\phi(w)\rangle &= -\log|z-w||\bar{z}-w|, \\ \langle\beta(z)\gamma(w)\rangle &= \frac{1}{w-z}, \\ \langle\bar{\beta}(\bar{z})\gamma(w)\rangle &= -\frac{1}{w-\bar{z}}, \\ \langle\beta(z)\bar{\gamma}(\bar{w})\rangle &= -\frac{1}{\bar{w}-z}, \\ \langle\bar{\beta}(\bar{z})\bar{\gamma}(\bar{w})\rangle &= \frac{1}{\bar{w}-\bar{z}}. \end{aligned} \tag{4.3}$$

These correlators are consistent with the fields β, γ being holomorphic, and the fields $\bar{\beta}, \bar{\gamma}$ being antiholomorphic, as implied by their respective bulk equations of motion in the free theory. They also agree with the gluing conditions, in the sense that

$$\lim_{z\rightarrow\bar{z}}\langle(\beta(z) + \bar{\beta}(\bar{z}))\cdots\rangle = \lim_{z\rightarrow\bar{z}}\langle(\gamma(z) + \bar{\gamma}(\bar{z}))\cdots\rangle = 0. \tag{4.4}$$

(For most purposes, the $\beta\gamma$ system with conformal weights $(1,0)$ is actually equivalent to a suitably normalized complex free boson ω such that $\beta = \partial\omega, \bar{\beta} = \bar{\partial}\omega, \gamma = \omega_L^*, \bar{\gamma} = \omega_R^*$,

where ω_L, ω_R are the holomorphic and antiholomorphic terms of ω respectively, and the star denotes complex conjugation.)

Let us check that the bulk and boundary interaction terms preserve the affine symmetries (2.5). Our treatment of this problem is inspired from [16], where more general results and references on perturbed boundary conformal field theories can be found. To first order in the boundary coupling constant c , the $J^+ + \bar{J}^+$ symmetry condition is

$$\lim_{z \rightarrow \bar{z}} \left\langle (J^+(z) + \bar{J}^+(\bar{z})) \int L^{\text{bdy}}(\tau) \dots \right\rangle = 0. \tag{4.5}$$

Notice that $\lim_{z \rightarrow \bar{z}} \langle (J^+(z) + \bar{J}^+(\bar{z})) L^{\text{bdy}}(\tau) \dots \rangle$ vanishes for all τ due to the symmetries of the free theory. The symmetry condition we just wrote might nevertheless fail because of the singularities which appear as the operators $J^+ + \bar{J}^+$ and L^{bdy} come close, and which might prevent the integration over τ from commuting with the limit $\lim_{z \rightarrow \bar{z}}$. Using the contractions (4.3), the singular terms coming from J^+ are²

$$J^+(z)L^{\text{bdy}}(\tau) \sim \frac{b^{-2}}{(z-\tau)^2} e^{b\phi(\tau)} + \frac{2b^{-1}}{z-\tau} \partial\phi(\tau) e^{b\phi(\tau)} = b^{-2} \partial_\tau \frac{e^{b\phi(\tau)}}{z-\tau} - \frac{i}{z-\tau} \partial_\sigma \phi(\tau) e^{b\phi(\tau)}. \tag{4.6}$$

The first term is a total τ -derivative and will therefore not contribute to the τ -integral. The second term vanishes due to the Neumann gluing condition on ϕ , namely $\partial_\sigma \phi = 0$. Similar calculations show that the singular terms coming from \bar{J}^+ also vanish, and therefore the symmetry condition (4.5) holds. The $J^3 - \bar{J}^3$ and $J^- + \bar{J}^-$ symmetry conditions can similarly be checked.

Of course, this is not enough for fully establishing the current symmetries of the theory. One would need to check that say $\lim_{z \rightarrow \bar{z}} (J^+(z) + \bar{J}^+(\bar{z}))$ vanishes when inserted into correlators with arbitrary numbers of insertions of both the bulk and the boundary interaction terms $\iint L^{\text{bulk}}, \int L^{\text{bdy}}$. The case with just one bulk interaction term can be treated along the same lines as above, but is a bit more tedious. We abstain from displaying such calculations, because our main aim is only to check the correctness of the boundary action.

5. Free-field calculation of the bulk one-point function

Let us demonstrate the validity of the free-field formalism by computing the bulk one-point function. The calculation is quite similar to the free-field calculation of the bulk one-point function in Liouville theory with Neumann boundary conditions, which was sketched in [17]. The new features of the H_3^+ case are the contribution of the $\beta\gamma$ system, and the resulting dependence of the one-point function on the isospin μ of the bulk field. In this section we will explicitly give the values of the relevant integrals over the worldsheet positions of the bulk and boundary interaction terms in the H_3^+ model Lagrangian, which might also be useful for other applications.

²Notice that $\bar{J}^+(\bar{z})L^{\text{bdy}}(\tau) \sim -\frac{b^{-2}}{(\bar{z}-\tau)^2} e^{b\phi(\tau)} - \frac{2b^{-1}}{\bar{z}-\tau} \partial\phi(\tau) e^{b\phi(\tau)}$, which is in accordance with the vanishing of $\lim_{z \rightarrow \bar{z}} \langle (J^+(z) + \bar{J}^+(\bar{z})) L^{\text{bdy}}(\tau) \dots \rangle$ for any given τ .

The one-point function of a bulk field $\Phi^j(\mu|z)$ (3.1) in the presence of an AdS_2 D-brane with parameter c is expected to be amenable to a free-field calculation only for certain quantized values of the spin j , namely

$$2j + 1 = -n \in -\mathbb{N} . \tag{5.1}$$

The one-point function actually has simple poles at such values of the spin j , whose residues are expressed as

$$\begin{aligned} \text{Res}_{2j+1=-n} \langle \Phi^j(\mu|z) \rangle_c^{H_3^+} &= \frac{1}{2b} |\mu|^{2j+2} \sum_{\substack{m, \ell=0 \\ 2m+\ell=n}}^{\infty} \frac{1}{m! \ell!} \prod_{i=1}^m \iint d^2 w_i \prod_{k=1}^{\ell} \int dx_k \\ &\left\langle e^{\mu\gamma(z) - \bar{\mu}\bar{\gamma}(\bar{z})} e^{2b(j+1)\phi(z, \bar{z})} \prod_{i=1}^m \lambda \frac{b^2}{2\pi} \beta \bar{\beta} e^{2b\phi}(w_i) \prod_{k=1}^{\ell} \frac{ic}{4\pi} \beta e^{b\phi}(x_k) \right\rangle , \end{aligned} \tag{5.2}$$

where the correlator on the second line is computed in the free theory described in the previous section.

This correlator factorizes into independent $\beta\gamma$ and ϕ correlators. Remembering that the field ϕ has a background charge b , the ϕ correlator is non-vanishing provided $2j + 1 + 2m + \ell = 0$; this is the origin of the condition (5.1). The ϕ correlator is then

$$\begin{aligned} \left\langle e^{2b(j+1)\phi(iy)} \prod_{i=1}^m e^{2b\phi(w_i)} \prod_{k=1}^{\ell} e^{b\phi(x_k)} \right\rangle &= \left[\prod_{k=1}^{\ell} (y^2 + x_k^2) \prod_{i=1}^m |y^2 + w_i^2|^2 \right]^{b^2(n-1)} \\ &\times |2y|^{-\frac{b^2}{2}(n-1)^2} \times \left[\prod_{i,k} |w_i - x_k|^2 \prod_{i < i'} |w_i - w_{i'}|^2 \prod_{i, i'} |w_i - \bar{w}_{i'}| \prod_{k < k'} |x_k - x_{k'}| \right]^{-2b^2} . \end{aligned} \tag{5.3}$$

In the $\beta\gamma$ correlator, integrating over the zero-modes of the field γ yields a factor $2\pi\delta(\mu + \bar{\mu})$. The rest of the computation follows from the contractions (4.3), which in particular imply

$$\left\langle e^{\mu\gamma(z) - \bar{\mu}\bar{\gamma}(\bar{z})} \beta(w) \right\rangle = \frac{\mu}{w - z} + \frac{\bar{\mu}}{w - \bar{z}} = \frac{\mu(z - \bar{z})}{(w - z)(w - \bar{z})} . \tag{5.4}$$

The actual correlator is a product of such factors,

$$\begin{aligned} &\left\langle e^{\mu\gamma(iy) - \bar{\mu}\bar{\gamma}(-iy)} \prod_{i=1}^m \lambda \frac{b^2}{2\pi} \beta \bar{\beta}(w_i) \prod_{k=1}^{\ell} \frac{ic}{4\pi} \beta(x_k) \right\rangle \\ &= 2\pi\delta(\mu + \bar{\mu})(-)^m \left(\lambda \frac{b^2}{2\pi} \right)^m |2y\mu|^n \left(-\frac{ic}{4\pi} \text{sgn}\Im\mu \right)^{\ell} \prod_{i=1}^m \frac{1}{|y^2 + w_i^2|^2} \prod_{k=1}^{\ell} \frac{1}{y^2 + x_k^2} . \end{aligned} \tag{5.5}$$

It is already clear that, after combining the factors (5.3) and (5.5), the remainder of the calculation is identical to the free-field calculation of the one-point function $\langle e^{2\alpha\phi(iy)} \rangle_{\mu_B}^{\text{Liouville}}$ of a Liouville field with parameter $\alpha = b(j+1) + \frac{1}{2b}$, in the presence of a Neumann boundary

with cosmological constant $\mu_B = c \frac{i}{4\pi} \text{sgn} \Im \mu$. The integral in eq. (5.2) amounts to taking the special value $a = 1 + b^2 - b^2 n$ in the following integral:

$$\mathcal{J}_{n,m}(a|y) = \frac{1}{m!(n-2m)!} \iint \prod_{i=1}^m \frac{d^2 w_i}{|y^2 + w_i^2|^{2a}} \int \prod_{k=1}^{n-2m} \frac{dx_k}{(y^2 + x_k^2)^a} \left[\prod_{i,k} |w_i - x_k|^2 \prod_{i < i'} |w_i - w_{i'}|^2 \prod_{i,i'} |w_i - \bar{w}_{i'}| \prod_{k < k'} |x_k - x_{k'}| \right]^{-2b^2}, \quad (5.6)$$

where as before the double integrals are over the upper half-plane, and the single integrals over the real line. This integral can be evaluated explicitly. With the notations $s(x) \equiv \sin \pi x$ and $C_n^i = \frac{n!}{i!(n-i)!}$, the result is

$$\mathcal{J}_{n,m}(a|y) = \frac{|2y|^{n(1-2a-(n-1)b^2)}}{n!} \left(\frac{2\pi}{\Gamma(1-b^2)} \right)^n \frac{2^{-2m}}{(s(b^2))^m} I_n(a) J_{n,m}(a), \quad (5.7)$$

where

$$I_n(a) = \prod_{i=0}^{n-1} \frac{\Gamma(1-(i+1)b^2)\Gamma(2a-1+(n-1+i)b^2)}{\Gamma^2(a+ib^2)}, \quad (5.8)$$

and

$$J_{n,m}(a) = \sum_{i=0}^m (-)^i C_{n-m-i}^{m-i} \frac{s((n+1-2i)b^2)}{s((n+1-i)b^2)} \prod_{r=0}^{i-1} \frac{s((n-r)b^2)s(a+(n-r)b^2)}{s((r+1)b^2)s(a+rb^2)}. \quad (5.9)$$

Let us introduce new notations for the boundary parameter c :

$$c = -\sqrt{\lambda \frac{8\pi b^2}{\sin \pi b^2}} \sinh r = -i \text{sgn} \Im \mu \sqrt{\lambda \frac{8\pi b^2}{\sin \pi b^2}} \cosh \sigma, \quad (5.10)$$

$$\text{where } \sigma = r - i \frac{\pi}{2} \text{sgn} \Im \mu. \quad (5.11)$$

We now sum over the numbers of screening charges m, ℓ while keeping $2m + \ell = n$ fixed like in eq. (5.2). This summation reduces to the following formula, which can be proved by the application of standard trigonometric identities:

$$\sum_{m=0}^{\lfloor n/2 \rfloor} (-)^m (2 \cosh \sigma)^{n-2m} J_{n,m}(a) = \sum_{i=0}^n \cosh((n-2i)\sigma) \frac{[-nb^2]_i [-(n-1)b^2 - a]_i}{[b^2]_i [a]_i}, \quad (5.12)$$

where we use the notation $[x]_i \equiv \prod_{r=0}^{i-1} s(x + rb^2)$.

Actually, our calculation involves a relatively simple case of these formulas, because it only uses particular values of the parameter a , namely the values $a = 1 + b^2 - b^2 n$ with $n \in \mathbb{N}$. In this case, only the term $i = 0$ survives in eq. (5.12). The bulk-boundary two-point functions in Liouville theory and in H_3^+ would involve the general case. And the integral (5.6) might be useful for other applications.

The result of the free-field calculation is therefore:

$$\begin{aligned} \text{Res}_{2j+1=-n} \langle \Phi^j(\mu|iy) \rangle_c^{H_3^+} &= |2y|^{\frac{b^2}{2}(n^2-1)} \delta(\mu + \bar{\mu}) |\mu| \times \pi b^{-1} \left(\frac{2 \Gamma(1-b^2)}{\lambda \Gamma(1+b^2)} \right)^{-\frac{n}{2}} \\ &\times \frac{(-)^n}{n!} \Gamma(1-b^2 n) \cosh n \left(r - i \frac{\pi}{2} \text{sgn} \Im \mu \right), \end{aligned} \quad (5.13)$$

This should be compared to the conformal bootstrap result [6, 7], which in the notations of [18] reads:

$$\begin{aligned} \langle \Phi^j(\mu|iy) \rangle_c^{H_3^+} &= |2y|^{2b^2 j(j+1)} \delta(\mu + \bar{\mu}) |\mu| \times \pi (8b^2)^{-\frac{1}{4}} \left(\pi \frac{\Gamma(1-b^2)}{\Gamma(1+b^2)} \right)^{j+\frac{1}{2}} \\ &\times \Gamma(2j+1) \Gamma(1+b^2(2j+1)) \cosh(2j+1) \left(r - i \frac{\pi}{2} \text{sgn} \Im \mu \right). \end{aligned} \quad (5.14)$$

We thus find agreement, provided the bulk cosmological constant is chosen as $\lambda = \frac{2}{\pi}$, so that the normalizations of the bulk interaction term agree. The remaining discrepancy is just an overall numerical factor. The numerical normalization factor in eq. (5.14) was derived in [6] by a “modular bootstrap” calculation of the annulus amplitude, whereas we did not impose such a normalization here.

This agreement provides another test of the proposed boundary action of the H_3^+ model.

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